0.1 Squares Analysis Method S.O.S

0.1.1 The Begining Problems

Generally, if we have an usual inequalities, the ways for us to solve them are neither trying to fumble from well-known inequalities nor finding a mixing-variable method, the best choise is often bringing them back to squaring forms. That is based on the most elementary probperty of a real number : $x^2 \geq 0\ \forall x \in R$. There are alot of problems, though you are indeliberate or not, almost used this property in proving. However, what you will read hereafter is likely make you truthly suprising.

We will start with the inequality $AM - GM$, which can be see as the most well-known inequality of all comom inqualities. But we only consider it in some simply case of n. For example, if $n = 2$ we have

Example 0.1.1. *Prove that for all $a, b \geq 0$ we have the inequality*

$$a^2 + b^2 \geq 2ab$$

There are not many thing to mention in this one, even this is the first time you see a inequality, the solution is very easy. The inequality is equal to $(a - b)^2 \geq 0$, obviously. Now, consider it if $n = 3$, we have

Example 0.1.2. *Prove that for all $a, b, c \geq 0$ we have*

$$a^3 + b^3 + c^3 \geq 3abc$$

If asked about one demonstration for it, we fell a bit puzzled. Of course, it's not difficult, the solution is only in one-line

$$VT - VP = \frac{1}{2}(a + b + c)\ \big(\ (a - b)^2 + (b - c)^2 + (c - a)^2\ \big)$$

And definitally, this is the sharpest key, because we don't need to use any intermediate stages. Both examples can be proved by Squares analysis method, but in confined meaning. The special advantage of this method is using so little advance knowledge, even you don't need to know any inequality theorems. Moreover, this is still a natural way with our thought.

If you tried to read the problems in previous chapter carefully, you won't find problems using this method rarely. Now, by general the way using and find the essence of a extremely useful method.

An important problem which we consider cautious is a very nice and famous, that was introduced in previous chapter, Iran 96 inequality

Inequal* 1 (Iran 98). *Prove that for all $a, b, c \geq 0$ we have*

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \geq \frac{9}{4(ab+bc+ca)}$$

This is a problem which exists in a very simply and nice form. In addition, it is also very difficult if you haven't ever seen before. But the first, we will be concerned with the inequality appeared as problem A3 in the International Math Olympiad and find a really natural proving for it.

Example 0.1.3 (IMO 2005 Pro. A3). *Suppose that x, y, z are reals number and $xyz \geq 1$. Prove the following inequality*

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + y^2 + x^2} \geq 0$$

Solution. First, we rewrite it in standard- equal degree form

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \frac{x^5 - x^2.xyz}{x^5 + (y^2 + z^2)xyz} = \frac{x^4 - x^2yz}{x^4 + yz(y^2 + z^2)}$$
$$\frac{x^4 - x^2yz}{x^4 + yz(y^2 + z^2)} \geq \frac{2x^4 - x^2(y^2 + z^2)}{2x^4 + (y^2 + z^2)}$$

Let $a = x^2, b = y^2, c = z^2$, we need to prove

$$\sum_{a,b,c} \frac{2c^2 - a(b+c)}{2a^2 + (b+c)^2} \geq 0$$
$$\Leftrightarrow \sum_{a,b,c} (a-b)\frac{a}{2a^2 + (b+c)^2} - \frac{b}{2b^2 + (a+c)^2}) \geq 0$$
$$\Leftrightarrow \sum_{a,b,c} (a-b)^2 \frac{c^2 + c(a+b) + a^2 - ab + b^2}{(2a^2 + (b+c)^2)(2b^2 + (a+c)^2)} \geq 0$$

Which is obvious true. The equal holds if $a = b = c = 1$.
This is not the uequal solution for the inequality, maybe there're some nicer, but the most importance of that is giving us a very natual mothod with 3-variable inequality. Generally, if we have an abiraty inequality, try to rewrite it to the form

$$S_c(a-b)^2 + S_b(a-c)^2 + S_a(b-c)^2 \geq 0$$

Rewriting the inequality to this base-expression is the first step in the way using S.O.S. If you quitely familier with inequality then construction this expression is

relatively simple and easy, we only need some identity and transform. If not, I will explain for you in section "Basic-form of S.O.S method and some analysis techniques.

Of course, if in that basic-form, all confficinents S_a, S_b, S_c are non-negative, we are done. For a long time, this still only case for us but extremelly, it's only the first simply and easiet application of Squaring analysis thorem. More important, S.O.S help us to solve problems which we treated not to use yet : some of S_a, S_b, S_c are negative.

Generally, in some sysmetric we can assume that $a \geq b \geq c$. For the cyclic problem, we need to consider an extra case $c \geq b \geq a$. In case $a \geq b \geq c$ we have a comment

1. *If $S_b \geq 0$, because $(a-c)^2 \geq (a-b)^2 + (b-c)^2$, so we have*

   $$S_c(a-b)^2 + S_b(a-c)^2 + S_a(b-c)^2 \geq (S_c + S_b)(a-b)^2 + (S_b + S_a)(b-c)^2$$

   And the rest is proving $S_a + S_b \geq 0$, $S_c + S_b \geq 0$. But commonly, two those inequalities can be proved quite easy, because they haven't any square-expression $(a-b)^2, (b-c)^2, (c-a)^2$ yet.

2. *If $S_b \leq 0$, because $(a-c)^2 \leq 2(a-b)^2 + 2(b-c)^2$, so we have*

   $$S_c(a-b)^2 + S_b(a-c)^2 + S_a(b-c)^2 \geq (S_c + 2S_b)(a-b)^2 + (2S_b + S_a)(b-c)^2$$

   Also, proving $S_c + 2S_b \geq 0$ and $2S_b + S_a \geq 0$ is more simple.

In addition, we need some stronger assession. The usual assession is

$$\frac{a-c}{b-c} \geq \frac{a}{b}(a \geq b \geq c)$$

From that, if $S_b, S_c \geq 0$ then

$$S_b(a-c)^2 + S_a(b-c)^2 = (b-c)^2 \left(S_b(\frac{a-c}{b-c})^2 + S_a \right) \geq (b-c)^2(\frac{a^2 S_b}{b^2} + S_a)$$

And we are done if we prove $a^2 S_b + b^2 S_a \geq 0$ completely. We combinate all result in one theorem as follow

Theorem 0.1 (S.O.S Theorem (Squaring Analysis Theorem)). *Consider the expression*

$$S = f(a,b,c) = S_a(b-c)^2 + S_b(a-c)^2 + S_c(a-b)^2$$

When S_a, S_b, S_c are functions of a, b, c.

1. *If* $S_a, S_b, S_c \geq 0$ *then* $S \geq 0$.
2. *If* $a \geq b \geq c$ *and* $S_b, S_b + S_c, S_b + S_a \geq 0$ *then* $S \geq 0$.
3. *If* $a \geq b \geq c$ *and* $S_a, S_c, S_a + 2S_b, S_c + 2S_b \geq 0$ *then* $S \geq 0$.
4. *If* $a \geq b \geq c$ *and* $S_c, S_a \geq 0, a^2S_b + b^2S_a \geq 0$ *then* $S \geq 0$.
5. *If* $S_a + S_b + S_c \geq 0$ *and* $S_aS_b + S_bS_c + S_cS_a \geq 0$ *then* $S \geq 0$.

The application 5th can be easily prove by quaratic creation.

Moreover, if $S \geq 0$ for all a, b, c, we must have $S_a + S_b|_{a=b} \geq 0$, $S_b + S_c|_{b=c} \geq 0$, $S_c + S_a|_{c=a} \geq 0$ ($S_a + S_b|_{a=b}$ mean that we consider the expression $S_a + S_b$ when $a = b$. For sysmetric inequalities, we have $S_a = S_b$ if $a = b$, so S_a must be non-negative. The simply comment is very important for problem which we have to find the best constant satisfy.

The theorem seems to be so important and if we say that It has a very strong effect for almost 3-variable inequality, It's really incredible. But in fact, S.O.S has done this work and this's very supricing. One question is gived that for what expression, we can tranform it to basic S.O.S form? The answer is that we can do it for all sysmetric or cyclic function $f(a, b, c)$ satisfy $f(a, a, a) = 0$, f can involve fractions, roots... See the proving in next section.

Now I will give you some examples to prove the strong effect of this method. If you can, please solve in anyway you had and compare

Example 0.1.4. *Prove the inequality if* $a, b, c \geq 0$

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{8abc}{(a + b)(b + c)(c + a)} \geq 2$$

Solution. Notice in 2 indentity

$$a^2 + b^2 + c^2 = \frac{1}{2}\left((a - b)^2 + (b - c)^2 + (c - a)^2 \right)$$
$$(a + b)(b + c)(c + a) - 8abc = a(b - c)^2 + b(c - a)^2 + c(a - b)^2$$

So, when we subtract 1 in each hand, we have an equality one

$$\frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{ab + bc + ca} \geq \frac{2c(a - b)^2 + 2b(a - c)^2 + 2a(b - c)^2}{(a + b)(b + c)(c + a)}$$

We find that

$$S_a = \frac{(a+b)(b+c)(c+a)}{ab+bc+ca} - 2a = b+c-a-\frac{abc}{ab+bc+ca}$$
$$S_b = \frac{(a+b)(b+c)(c+a)}{ab+bc+ca} - 2b = a+c-b-\frac{abc}{ab+bc+ca}$$
$$S_c = \frac{(a+b)(b+c)(c+a)}{ab+bc+ca} - 2c = a+b-c-\frac{abc}{ab+bc+ca}$$

Because of the sysmetric property, we can assume that $a \geq b \geq c$. Indded, $S_b \geq 0, S_c \geq 0$. Use the second criterion of S.O.S theorem, we only need to prove $S_a+S_b \geq 0$. But It's obvious because

$$S_a + S_b = 2c - \frac{2abc}{ab+bc+ca} = \frac{2c^2(a+b)}{ab+bc+ca} \geq 0$$

So, we are done. The equal takes if $a=b=c$ or $a=b, c=0$ or all permutations .

Now we return to Iran 96 Inequality

Example 0.1.5 (Iran TST 1996). *Prove that for all $x, y, z \geq 0$*

$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \geq \frac{9}{4(xy+yz+zx)}$$

Solution. Let $a = x+y, b = y+z, c = z+x$. We need to prove

$$(2ab+2bc+2ca-a^2-b^2-c^2)(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}) \geq \frac{9}{4}$$

By simpy expanding, we have a equality one

$$(\frac{2}{bc}-\frac{1}{a^2})(b-c)^2 + (\frac{2}{ca}-\frac{1}{b^2})(a-c)^2 + (\frac{2}{ab}-\frac{1}{c^2})(a-b)^2 \geq 0$$
$$S_a = \frac{2}{bc} - \frac{1}{a^2} \quad , \quad S_b = \frac{2}{ca} - \frac{1}{b^2} \quad , \quad S_c = \frac{2}{ab} - \frac{1}{c^2}$$

Suppose that $a \geq b \geq c$, so $S_a \geq 0$. Using the fourth criterion we have to prove $b^2 S_b + c^2 S_c \geq 0 \Leftrightarrow b^3 + c^3 \geq 2abc$, but it's obvious because

$$a \leq b+c \Rightarrow b^3+c^3 \geq bc(b+c) \geq abc$$

The equal takes if $a=b=c$ or $a=b, c=0$ or all permutations .

There's some other ways to prove Iran 96 Inequality, the usual is directly expanding and use *Schur* or *Muihard*. But you will agree with me that those solutions

have only one mean is that proving the inequality true in Math, not make any impression for anyone. Using S.O.S theorom is not only giving for us a nice and simple solution, but also bring a new vision in inequality. Moreover, this solution is obsolutely satisfy beauty sense of Math.

The squaring analysis method has been appeared in some ways in inequalities because It's very natural. But certainly It's the first time when the method have been named and treated as the standard solution with inequality. It brings for us a very useful and active thought solving inequality, which a short time ago, they are extremely hard. The Iran 96 Inequality is treated as the most basic of S.O.S method, although I named it after an older inequality.